The Stability Test for Symmetric Alpha-Stable Distributions

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***Abstract*—Symmetric alpha-stable distributions are a popular statistical model for heavy-tailed phenomena encountered in communications, radar, biomedicine, and econometrics. The use of the symmetric alpha stable model is often supported by empirical evidence, where qualitative criteria are used to judge the fit, leading to subjective decisions. Objective decisions can only be made through quantitative statistical tests. Here, a goodness-of-fit hypothesis test for symmetric alpha-stable distributions is developed based on their unique stability property. Critical values for the test are found using both asymptotic theory and from bootstrap estimates. Experiments show that the stability test, using bootstrap estimates of the critical values, is better able to discriminate between symmetric alpha stable distributions and other heavy-tailed distributions than classical tests such as the Kolmogorov–Smirnov test.**

***Index Terms*—Alpha stable, bootstrap, goodness-of-fit, heavy-tailed distributions, hypothesis tests.**

I. Introduction

HEAVY-tailed distributions describe a class of phenomena characterized by impulses or spikes. Such behavior may be attributed to outliers, contamination by impulsive noise, or the fundamental nature of the phenomena.

The probability density functions (pdfs) of heavy-tailed distributions possess tails that decay at a slower rate than those of a Gaussian pdf. As this definition admits an unlimited number of alternatives, it is important to determine which one of these provides the best description since the performance of any procedure rests on the model efficiently capturing the statistical characteristics of the observations.

The symmetric alpha stable $(S\alpha S)$ distribution has been used to model a wide variety of heavy-tailed phenomena with applications including econometrics [1], various forms of electromagnetic interference, [2]–[4], synthetic aperature radar [5], and shot noise [6]. Theoretical motivation for the $S\alpha S$ model is provided by results such as the generalized central limit theorem (CLT), which extends the CLT. The empirical evidence is weaker as it is based only on qualitative subjective tests, such as visually comparing the amplitude probability distributions of impulsive noise and $S\alpha S$ distributions [3]. Furthermore, it has been shown that $S\alpha S$ distributions do not always provide a good fit to observations [7]. This makes powerful quantitative statistical tests necessary in order to properly evaluate whether the $S\alpha S$ distribution is a suitable model. In this paper, a powerful statistical test for $S\alpha S$ distributions is proposed.

Existing statistical techniques for assessing goodness of fit such as the Kolmogorov–Smirnov test are not powerful at discriminating between $S\alpha S$ distributions and others with similar tail behavior such as the Student's t distribution; rather, it is more suited to detecting differences in location or symmetry. Recently, a statistical test for alpha stable (αS) distributions was proposed based on the maximum distance between the empirical characteristic function (cf) and the parametric cf evaluated at estimates of the distributional parameters [8]. This is an adaptation of a well-known cf-based goodness-of-fit test [9].

The proposed test is based on the unique stability property of αS[1] distributions that sums of independent and identically distributed (i.i.d) αS random variables follow the same αS distribution. The test is implemented by dividing the observations into segments and testing whether summing these segments changes the distributional parameters. The stability testing concept has been used in exploratory analysis but has never been implemented in a statistical framework. Both asymptotic and bootstrap techniques are used to determine critical values, the latter proving more powerful in detecting alternatives similar in distribution to $S\alpha S$.

This paper is structured as follows. Section II introduces $S\alpha S$ distributions. Section III introduces goodness-of-fit from the hypothesis testing viewpoint and reviews existing techniques. The stability test is developed in Section IV. Bootstrap and asymptotic methods for obtaining the null distributions of the test statistics are presented in Section V. Experiments evaluating the stability test are presented in Section VI followed by conclusions in Section VII.

II. Symmetric Alpha Stable Distributions

The theory of αS distributions is well developed since being initiated by Cauchy [11]. For detailed expositions, see [10], [12], [13], and the references therein. Applications of αS distributions to signal processing can be found in [3].

Several equivalent definitions for αS distributions exist. The two required here are presented for the $S\alpha S$ case.

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[1]A generalization of $S\alpha S$ distributions to the nonsymmetric case [10].

Definition 1: A random variable X follows a $S\alpha S$ distribution if there are parameters $0 \leq \alpha \leq 2$ and $c > 0$ such that its cf is of the form

$$\phi_X(\omega) = \exp\left(-|c\omega|^{\alpha}\right) \quad -\infty < \omega < \infty. \tag{1}$$

The $S\alpha S$ distribution is uniquely defined by its parameters (α, c). The characteristic exponent α measures how heavy-tailed the distribution is, the asymptotic decay rate of the pdf tails being proportional to $|x|^{-\alpha-1}$ [10]. The scale parameter c measures dispersion or spread. The existence of a simple expression for the cf is fortunate as the pdf is not expressible in closed form, except for the Gaussian $(\alpha = 2)$ and Cauchy $(\alpha = 1)$ distributions. For this reason, much of the theory has been developed in the cf domain.

Definition 2: A symmetric random variable X follows a $S\alpha S$ distribution if for any $n \geq 2$, there is a $C_n > 0$ such that

$$X_1 + X_2 + \cdots + X_n \stackrel{d}{=} C_n X \tag{2}$$

where $X_1, X_2, \cdots, X_n$ are independent copies of X, $\stackrel{d}{=}$ denotes equality in distribution [10], and by symmetric, it is meant that X and $-X$ have the same distribution.

This definition demonstrates the stability property of $S\alpha S$ distributions; sums of i.i.d $S\alpha S$ random variables follow the same $S\alpha S$ distribution to within a scale factor C_n.

The $S\alpha S$ distribution possesses some interesting properties such as nonexistence of moments of order α or more for non-Gaussian $S\alpha S$ distributions, rendering most second- and higher order techniques unusable. Perhaps the most important property of αS distributions with respect to its use as a model is the generalized CLT. Briefly, it states that regardless of the existence of the variance, the limiting distribution of a sum of i.i.d random variables is αS. The generalized CLT reduces to CLT when the variance is finite, in which case, the limiting distribution is Gaussian. Just as the CLT is a powerful motivation for the use of a Gaussian model, so too does the generalized CLT provide motivation for αS models.

III. Goodness-of-Fit Tests

Given N i.i.d random variables $X_1, \ldots, X_N$ with unspecified cumulative distribution function (cdf) $G_X(x)$, a goodness-of-fit test decides if they follow the null distribution $F_X(x; \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ contains possibly unknown distributional parameters. The hypothesis testing structure is

$$\mathsf{H} : G_X(x) = F_X(x; \boldsymbol{\theta}) \tag{3a}$$

$$\mathsf{K} : G_X(x) \neq F_X(x; \boldsymbol{\theta}) \tag{3b}$$

where H is the null hypothesis, and K is the alternative hypothesis. In general, $\boldsymbol{\theta}$ is either known or replaced with suitable estimates. Here, the null distribution is $S\alpha S$.

Only Neyman–Pearson tests, where the probability of incorrectly accepting the alternative hypothesis is constrained, are considered here. The former probability is known as the probability of false alarm P_{FA}, and the constraint is $P_{\mathrm{FA}} \leq \zeta$, where the set level ζ is set *a priori*. A test meeting this constraint is said to maintain the set level; if $P_{\mathrm{FA}} < \zeta$, the test is said to be conservative. The power of the test is the probability of correctly accepting the alternative hypothesis, and a test is said to be powerful if its power is large compared to other tests. Given that the set level is maintained, goodness-of-fit tests are evaluated by comparing their powers.

Several techniques for goodness-of-fit testing exist. Specialized techniques exploit particular properties of the null distribution while generic methods include probability plots, χ^2, and empirical distribution function tests. Recent cf domain techniques that parallel empirical distribution function tests have also been proposed. For completeness, a brief overview follows with special regard to testing for $S\alpha S$ distributions.

A. Specialized Goodness-of-Fit Tests

Early tests for αS distributions utilized their algebraic tail behavior. On a log-log scale, the αS cdf is asymptotically linear in the tails and a visual judgement was made on whether the empirical cdf possessed this property [14]. Difficulties associated with this approach include heteroscedasticity and correlation in the plots and the large or infinite variance of extreme points. The region over which the tails of an empirical cdf follow this behavior closely enough for such a test to be successfully employed has not been definitively answered. However, some guidelines as to how far out in the tails one must be for this behavior to become dominant are given in [15].

The lack of a finite variance for non-Gaussian αS distributions leads to a test based on the running sample variance of the observations. For distributions with finite variance, a plot of sample variance versus the number of observations used will converge to its true value by the law of large numbers but will diverge for distributions with infinite variance. A subjective judgement can be made to ascertain convergence.

The application of the above two tests is simple, though they are subjective, not powerful, and exclude the Gaussian distribution. They also admit non-$S\alpha S$ distributions under the null hypothesis, where the test for algebraic tails test admits any distribution with algebraic tail decay and the test for infinite variance admits all those with infinite variance. An example of a non-$S\alpha S$ distribution with both these properties is Student's t distribution with two degrees of freedom.

B. Generic Goodness-of-Fit Tests

The previous tests are specific to αS and other heavy-tailed distributions. Generic goodness-of-fit tests include probability plots, χ^2, empirical distribution function, and empirical cf tests.

Probability plots are graphical tests useful in exploratory work. They plot a transformation of the empirical cdf such that under the hypothesised distribution a straight line is obtained. A plot of the empirical percentiles versus percentiles under the null hypothesis are best when testing for heavy-tailed distributions; quantiles are less effective. Regression tests or human judgement are then used to measure the fit [16]. Probability plots remain subjective at the added cost of evaluating the cdf and estimating distributional parameters, and the evaluation of $S\alpha S$ cdfs is not trivial computationally.

χ^2 squared tests group the observations into bins and compare observed with expected counts under the null hypothesis. For a fully specified null, asymptotic theory provides critical values. Corrections to the classical χ^2 statistic can be found for the case where the distributional parameters must be estimated, as can guidelines for selecting the bins, although not specifically for the αS case [17]. Although χ^2 tests are quantitative, disadvantages include evaluation of the cdf and a loss of information from grouping the observations.

Empirical distribution function tests measure the distance between the empirical and null cdfs. Supremum and quadratic measures are commonly used [17]. Supremum statistics, including the well-known Kolmogorov–Smirnov statistic, measure vertical differences

$$D = \sup_x \left| \hat{G}_X(x) - F_X(x; \boldsymbol{\theta}) \right| \tag{4}$$

where $\hat{G}_X(x)$ is the empirical cdf of the observations. Quadratic statistics, which are also known as the Cramér-von-Mises family, measure the weighted integrated squared error between the empirical and null cdfs

$$Q = N \int_{-\infty}^{\infty} \left(\hat{G}_X(x) - F_X(x; \boldsymbol{\theta}) \right)^2 w(x) dF_X(x; \boldsymbol{\theta}). \tag{5}$$

Different weighting functions $w(x)$ yield the various quadratic statistics including the Cramér-von-Mises statistic W^2 when $w(x) = 1$ and the Anderson–Darling statistic A^2 when $w(x) = (F_X(x; \boldsymbol{\theta})(1 - F_X(x; \boldsymbol{\theta}))^{-1}$.

Evaluation of the test statistics are simplified by using the probability integral transform $F_X(X_n; \boldsymbol{\theta})$ to transform the observations to a uniform distribution on [0, 1], $U(0, 1)$, under the null hypothesis. The general problem then becomes one of testing for a uniform distribution, and simple expressions for the test statistics exist in this case [17].

The asymptotic null distributions of these statistics are known for a fully specified null hypothesis while corrections are available for finite samples. When distributional parameters are estimated, the asymptotic distributions depend on quantities such as $F_X(x; \boldsymbol{\theta})$, the true parameters, the estimator, and the sample size. Critical values must then be computed through Monte Carlo simulation [17].

In general, W^2 and A^2 tend to be more powerful than other empirical distribution function statistics such as D. Compared to W^2, A^2 gives more weight to extreme observations by virtue of $w(x)$ and tends to be more powerful at detecting departures in the tails [17]. This makes it more suited to testing for heavy-tailed distributions.

Empirical cf tests measure the distance between the empirical cf $\hat{\phi}_X(\omega)$ and the null cf $\phi_X(\omega)$. On this basis, they are similar to empirical distribution function tests, and hence, similar measures of distance are used. Kolmogorov–Smirnov type measures can be developed

$$\sup_\omega \left| \hat{\phi}_X(\omega) - \phi_X(\omega) \right| \tag{6}$$

as can Cramér-von-Mises type measures

$$\int_{-\infty}^{\infty} \left| \hat{\phi}_X(\omega) - \phi_X(\omega) \right|^2 v(\omega) d\omega \tag{7}$$

with $v(\omega)$ being a weighting function in the cf domain [17]. The former was used to test for the Gaussian distribution [9], whereas a quadratic statistic was used in [18]. Supremum-based tests for αS distributions have also been proposed where the bootstrap was used to estimate critical values, avoiding more extensive Monte Carlo simulations [8].

Advantages of empirical cf goodness-of-fit tests in this problem includes the mathematical tractability of the $S\alpha S$ cf, that the cf completely characterizes a random variable and favorable properties of the empirical cf such as strong consistency and asymptotic normality [12], [19], [20].

The more powerful statistical tests among those mentioned are of the empirical distribution function and empirical cf type. While emperical cf tests avoid costly evaluation of the cdf, determination of critical values is a computational burden for both.

IV. Stability Test

The stability property of Definition 2 will be exploited to test for $S\alpha S$ distributions. The concept is to split the sample into a number of nonoverlapping segments and then to estimate the characteristic exponent when the segments are summed elementwise. Under the null hypothesis, the characteristic exponent is invariant to how many segments are summed, given statistical variations caused by a finite sample.

In the statistical literature, the stability test was suggested as an exploratory technique [14], [21], [22], where the characteristic exponents from sums of two to ten segments were compared subjectively [23]–[26]. Here, the stability test is developed as a formal statistical hypothesis test.

Practical guidelines for the number of segments required to implement the stability test do not exist. Direct application of Definition 2 suggests the number of segments be at least 2 but otherwise unbounded. The following result avoids this impractical solution [11].

Property 1: It is not necessary to ensure that (2) holds for all $n \geq 2$; rather, a necessary and sufficient condition for X to have a $S\alpha S$ distribution is if

$$X_1 + X_2 \stackrel{d}{=} C_2 X \tag{8a}$$

$$X_1 + X_2 + X_3 \stackrel{d}{=} C_3 X \tag{8b}$$

where $C_2, C_3 > 0$, and X_1, X_2, and X_3 are independent copies of X. That is, Definition 2 needs only be confirmed for $n = 2$ and $n = 3$.

Clearly, this property can also be regarded as a definition for αS distributions. Although this settles the minimum number of segments required, using more may increase the power of the test, and this is discussed later.

Since characteristic exponents from sums of two and three nonoverlapping segments are necessary to confirm stability, define

$$Z_2 = X_1 + X_2 \tag{9a}$$

$$Z_3 = X_1 + X_2 + X_3. \tag{9b}$$

1^{st} segment				2^{nd} segment					M^{th} segment			
$X_{1M}(1)$	$X_{1M}(2)$	$\cdots$	$X_{1M}(L)$	$X_{2M}(1)$	$X_{2M}(2)$	$\cdots$	$X_{2M}(L)$		$X_{MM}(1)$	$X_{MM}(2)$	$\cdots$	$X_{MM}(L)$

$\underbrace{\qquad\qquad\qquad\qquad}_{L \text{ observations}}$

Fig. 1. Separation of the observations into the M segments, each segment being of length L.

It follows that $\alpha_X = \alpha_{Z_2}$ and $\alpha_X = \alpha_{Z_3}$ under the null hypothesis, and the hypothesis test is formulated as

$$\mathsf{H} : \alpha_{Z_2} = \alpha_X \cap \alpha_{Z_3} = \alpha_X \tag{10a}$$
$$\mathsf{K} : \alpha_{Z_2} \neq \alpha_X \cup \alpha_{Z_3} \neq \alpha_X. \tag{10b}$$

By convention, the intersection of the two null hypotheses $\alpha_X = \alpha_{Z_2}$ and $\alpha_X = \alpha_{Z_3}$ is referred to as the global null hypothesis. These hypotheses suggest the test statistics $T_2 = \alpha_{Z_2} - \alpha_X$ and $T_3 = \alpha_{Z_3} - \alpha_X$ so that the global null hypothesis is only accepted if both the hypotheses $T_2 = 0$ and $T_3 = 0$ are accepted.

To maintain a global level of significance when the global null hypothesis is comprised of more than one hypothesis, a multiple hypothesis test (MHT) procedure is used [27]. MHTs are easily described in terms of the ordered p-values $\mathcal{P}_{(1)} \leq \mathcal{P}_{(2)}$, where $\mathcal{P} = 1 - F_{\hat{T}}(\hat{T})$, and $F_{\hat{T}}$ is the cdf of the test statistic $\hat{T}$.

For independent hypotheses, Bonferroni's procedure exactly maintains the global level, accepting the global null hypothesis if $\mathcal{P}_{(1)} > \zeta/2$, where ζ is the global level of significance. For dependent hypotheses, this test becomes conservative and is less powerful [27]. Less conservative procedures include those of Holm, Hochburg, and Simes. As applied here, the procedures of Hochberg and Simes are simultaneously the least conservative and most powerful of those mentioned. Herein, only Hochberg's MHT is used as it accepts the global null hypothesis if $\mathcal{P}_{(1)} > \zeta/2$ and $\mathcal{P}_{(2)} > \zeta$, rejecting it otherwise. This procedure can be seen to be less conservative than Bonferroni's due to the extra condition $\mathcal{P}_{(2)} > \zeta$.

A. Development of the Test Statistics

To evaluate the test statistics and implement the stability test, several independent realizations from the distribution that generated the observations are required. To obtain these independent realizations, the observations are separated into segments.

There are then two ways the test statistics can be obtained. Consider the statistic T_2.

1) Separate the observations into three equilength segments. Arbitrarily assign the first segment to X, from which α_X is found. Sum the other segments elementwise to create Z_2, from which α_{Z_2} is found.
2) Assign all the observations to X so that α_X is found from the whole sample. Separate the observations into two equilength segments and sum them elementwise to obtain Z_2, from which α_{Z_2} is found. Note that X and Z_2 are of different lengths.

The first approach ensures that X, X_1, and X_2 are i.i.d. The second introduces dependence between X and X_1, X_2. The resulting dependence between $\hat{\alpha}_X$ and $\hat{\alpha}_{Z_2}$ confounds theoretical analysis and may reduce the power of the test since the estimates will be correlated, reducing their difference under the alternative. However, the reduced variance of $\hat{\alpha}_X$ may counteract this and, as will be shown, results in a more powerful test. The same comments apply to T_3.

Now, let $X_{mM}(l) = X((m-1)L + l), l = 1, \ldots, L$ denote segment m after the i.i.d observations $X(n), n = 1, \ldots, N$ have been split into M segments of length $L = \lfloor N/M \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part. The separation of the observations into the M segments is shown in Fig. 1. Define Z_{kM} as the elementwise sum of k segments from M. The following arbitrary ordering is used:

$$\begin{aligned}
Z_{22}(l) &= X_{12}(l) + X_{22}(l) \\
Z_{13}(l) &= X_{13}(l) \\
Z_{23}(l) &= X_{23}(l) + X_{33}(l) \\
Z_{33}(l) &= X_{13}(l) + X_{23}(l) + X_{33}(l) \\
Z_{14}(l) &= X_{14}(l) \\
Z_{34}(l) &= X_{24}(l) + X_{34}(l) + X_{44}(l), \quad l = 1, \ldots, L.
\end{aligned} \tag{11}$$

The two ways of constructing the global null hypothesis are then

$$T_{22} = \alpha_{Z_{22}} - \alpha_X = 0 \quad \cap \quad T_{33} = \alpha_{Z_{33}} - \alpha_X = 0 \tag{12a}$$
$$T_{23} = \alpha_{Z_{23}} - \alpha_{Z_{13}} = 0 \quad \cap \quad T_{34} = \alpha_{Z_{34}} - \alpha_{Z_{14}} = 0. \tag{12b}$$

In (12a), the characteristic exponents forming the test statistics are dependent, whereas in (12b), they are independent.

Characteristic exponents are estimated using the method of Koutrouvelis [28]. Numerous other methods exist including maximum likelihood [29], Bayesian [30], fractional lower order moments [31], and sample fractile and order statistics estimators [23], not to mention several refined versions of Koutrouvelis' original procedure [32]. Koutrouvelis' method was chosen as it offers a good compromise between computational cost and performance in comparison to most other methods, whereas the refinements mentioned offer improved performance mainly for nonsymmetric αS distributions, which are not of concern here.

V. Evaluation of Critical Values

Critical values or, equivalently, p-values of the test will be found in two ways. In the first, the asymptotic distributions of the statistics are derived. Since the test statistics (12a) and (12b) are the difference between two characteristic exponents, the asymptotic distributions of Koutrouvelis' estimator for the characteristic exponent will be found. For finite samples, asymptotic results may not always be applicable. Unfortunately, the finite sample distributions are nontrivial, making an analytical approach prohibitively complex. The second approach is then to estimate the finite sample distributions using a computational method known as the bootstrap.

A. Asymptotic Theory

Koutrouvelis' estimator is based on the cf domain representation for a $S\alpha S$ random variable (1). Applying the transformation $\log(-\log \mathrm{Re}(\cdot)^2)$ and using the empirical cf $\hat{\phi}_X(\omega)$ as an estimate for $\phi_X(\omega)$, α and c are estimated through a linear regression over several points ω_k, $k = 1, \ldots, K$, in the cf domain

$$\log\left(-\log \mathrm{Re}\left(\hat{\phi}_X(\omega_k)\right)^2\right) = \log(2c^\alpha) + \alpha \log|\omega_k|. \quad (13)$$

Since $\hat{\phi}_X(\omega)$ converges to $\phi_X(\omega)$ with probability one [20] and the parameters of a $S\alpha S$ distribution are unique, both $\hat{\alpha}$ and $\hat{c}$ are strongly consistent. The real part of the empirical cf is taken because the $S\alpha S$ cf is purely real. Koutrouvelis had considered αS distributions for which the cf is generally complex and used the absolute value of the empirical cf. As the asymptotic ($\omega \to \infty$) variance of $\mathrm{Re}(\hat{\phi}_X(\omega))^2$ is half that of $|\hat{\phi}_X(\omega)|^2$, this does not negatively impact on estimation.

Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_K)^\mathsf{T}$, $\hat{\boldsymbol{\phi}}_X = (\hat{\phi}_X(\omega_1), \ldots, \hat{\phi}_X(\omega_K))^\mathsf{T}$, and $\mathbf{1}_K$ be a K-length vector of ones, where $(\cdot)^\mathsf{T}$ denotes transposition. The regression (13) can then be written as

$$\boldsymbol{u} = \boldsymbol{V}\boldsymbol{\theta} + \boldsymbol{\epsilon} \quad (14)$$

where $\boldsymbol{u} = \log(-\log \mathrm{Re}(\hat{\boldsymbol{\phi}}_X)^2)$, $\boldsymbol{V} = (\mathbf{1}_K, \log|\boldsymbol{\omega}|)$, $\boldsymbol{\theta} = (\log(2c^\alpha), \alpha)^\mathsf{T}$, and $\boldsymbol{\epsilon}$ is a K length vector of correlated disturbances. The least squares solution for $\hat{\boldsymbol{\theta}}$ is

$$\hat{\boldsymbol{\theta}} = \boldsymbol{V}^\dagger \boldsymbol{u} \quad (15)$$

where $\boldsymbol{V}^\dagger = (\boldsymbol{V}^\mathsf{T}\boldsymbol{V})^{-1}\boldsymbol{V}^\mathsf{T}$ denotes the Moore–Penrose pseudoinverse of $\boldsymbol{V}$.

The asymptotic distribution of $\hat{\boldsymbol{\theta}}$ is obtained in three steps: Find the asymptotic distribution of $\hat{\boldsymbol{\phi}}_X$, apply the nonlinear transform $\log(-\log \mathrm{Re}(\cdot)^2)$, then apply the linear transform $\boldsymbol{V}^\dagger$.

From the multidimensional CLT [33], it follows that $\mathrm{Re}(\hat{\boldsymbol{\phi}}_X) \overset{a}{\sim} \mathrm{MVN}(\boldsymbol{\mu}_{\mathrm{Re}\boldsymbol{\phi}}, \boldsymbol{R}_{\mathrm{Re}\boldsymbol{\phi}})$ [20], [34], where $\overset{a}{\sim}$ denotes asymptotically distributed as; this is shown in (16a) and (16b) at the bottom of the page. $(\boldsymbol{\mu}_{\mathrm{Re}\boldsymbol{\phi}})_i$ denotes element i of vector $\boldsymbol{\mu}_{\mathrm{Re}\boldsymbol{\phi}}$, $(\boldsymbol{R}_{\mathrm{Re}\boldsymbol{\phi}})_{ij}$ denotes element (i,j) of matrix $\boldsymbol{R}_{\mathrm{Re}\boldsymbol{\phi}}$, and $\mathrm{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes a multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$.

The following theorem [33, Th. 3.3A] regarding functions of asymptotically MVN random vectors is needed for the nonlinear transformation,

Theorem 1: Given that $\boldsymbol{X} = (X_1, \ldots, X_K)^\mathsf{T} \overset{a}{\sim} \mathrm{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ such that $\boldsymbol{\Sigma} \to 0$ as $N \to \infty$, let $\boldsymbol{g}(\boldsymbol{x}) = (g_1(\boldsymbol{x}), \ldots, g_M(\boldsymbol{x}))^\mathsf{T}$, where every g_m is a vector valued function with nonzero derivative at $\boldsymbol{x} = \boldsymbol{\mu}$. Define the Jacobian of the transformation at $\boldsymbol{x} = \boldsymbol{\mu}$ as

$$(\boldsymbol{D})_{km} = \left.\frac{\partial g_m}{\partial x_k}\right|_{\boldsymbol{x}=\boldsymbol{\mu}}. \quad (17)$$

Then

$$\boldsymbol{g}(\boldsymbol{x}) \overset{a}{\sim} \mathrm{MVN}\left(\boldsymbol{g}(\boldsymbol{\mu}), \boldsymbol{D}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{D}\right). \quad (18)$$

Define $g_m(\boldsymbol{x}) = g(x_m)$, where $g(x) = \log(-\log x^2)$. The partial derivatives $g'(x) = 1/(x \log|x|)$ are nonzero at $(\boldsymbol{\mu}_{\mathrm{Re}\boldsymbol{\phi}})_k$ for a $S\alpha S$ cf evaluated at $0 < \omega_k < \infty$, and it follows from the above theorem that since $\boldsymbol{R}_{\mathrm{Re}\boldsymbol{\phi}} \to 0$ as $N \to \infty$

$$\log\left(-\log \mathrm{Re}(\hat{\boldsymbol{\phi}}_X)^2\right) \overset{a}{\sim} \mathrm{MVN}(\boldsymbol{\mu}_g, \boldsymbol{R}_g) \quad (19)$$

where $\boldsymbol{\mu}_g = \boldsymbol{g}(\boldsymbol{\mu}_{\mathrm{Re}\boldsymbol{\phi}})$, $\boldsymbol{R}_g = \boldsymbol{D}_g^\mathsf{T}\boldsymbol{R}_{\mathrm{Re}\boldsymbol{\phi}}\boldsymbol{D}_g$, and $\boldsymbol{D}_g$ is a diagonal matrix with $(\boldsymbol{D}_g)_{kk} = g'((\boldsymbol{\mu}_{\mathrm{Re}\boldsymbol{\phi}})_k)$.

From (15), it can be seen that $\hat{\boldsymbol{\theta}}$ will be asymptotically MVN as it is a linear transformation of the asymptotically MVN $\boldsymbol{u}$

$$\hat{\boldsymbol{\theta}} \overset{a}{\sim} \mathrm{MVN}(\boldsymbol{\mu}_\theta, \boldsymbol{R}_\theta) \quad (20)$$

where $\boldsymbol{\mu}_\theta = \boldsymbol{V}^\dagger\boldsymbol{\mu}_g$, and $\boldsymbol{R}_\theta = \boldsymbol{V}^\dagger\boldsymbol{R}_g\boldsymbol{V}^{\dagger^\mathsf{T}}$.

The asymptotic joint distribution of $\hat{c}$ and $\hat{\alpha}$ is obtained from the transformation

$$\hat{c} = g_1(\hat{\boldsymbol{\theta}}) = \left(\frac{e^{\hat{\theta}_1}}{2}\right)^{\frac{1}{\hat{\theta}_2}} \quad (21a)$$

$$\hat{\alpha} = g_2(\hat{\boldsymbol{\theta}}) = \hat{\theta}_2 \quad (21b)$$

for which the Jacobian at $\boldsymbol{\mu}_\theta$ is

$$\boldsymbol{D} = \begin{pmatrix} \frac{1}{(\boldsymbol{\mu}_\theta)_2}\left(\frac{e^{(\boldsymbol{\mu}_\theta)_1}}{2}\right)^{\frac{1}{(\boldsymbol{\mu}_\theta)_2}} & 0 \\ -\frac{1}{(\boldsymbol{\mu}_\theta)_2}\left(\frac{e^{(\boldsymbol{\mu}_\theta)_1}}{2}\right)^{\frac{1}{(\boldsymbol{\mu}_\theta)_2}} \log\left(\frac{e^{(\boldsymbol{\mu}_\theta)_1}}{2}\right)^{\frac{1}{(\boldsymbol{\mu}_\theta)_2}} & 1 \end{pmatrix} \quad (22)$$

so that

$$\begin{pmatrix} \hat{c} \\ \hat{\alpha} \end{pmatrix} \overset{a}{\sim} \mathrm{MVN}(\boldsymbol{\mu}_{c\alpha}, \boldsymbol{R}_{c\alpha}) \quad (23)$$

where $\boldsymbol{\mu}_{c\alpha} = ((e^{(\boldsymbol{\mu}_\theta)_1}/2)^{1/(\boldsymbol{\mu}_\theta)_2}, (\boldsymbol{\mu}_\theta)_2)^\mathsf{T}$, and $\boldsymbol{R}_{c\alpha} = \boldsymbol{D}^\mathsf{T}\boldsymbol{R}_\theta\boldsymbol{D}$. The estimates $\hat{\alpha}$ and $\hat{c}$ are asymptotically normal and asymptotically unbiased. It was found that for $N > 200$, the bias is negligible.

The asymptotic distributions of the test statistics under the null hypothesis can now be found. Take the test statistic $\hat{T}_2 =$

$$(\boldsymbol{\mu}_{\mathrm{Re}\boldsymbol{\phi}})_i = \mathsf{E}\left[\mathrm{Re}\left(\hat{\phi}_X(\omega_i)\right)\right] = \mathrm{Re}\left(\phi_X(\omega_i)\right) \quad (16a)$$

$$\begin{aligned}(\boldsymbol{R}_{\mathrm{Re}\boldsymbol{\phi}})_{ij} &= \mathsf{Cov}\left[\mathrm{Re}\left(\hat{\phi}_X(\omega_i)\right), \mathrm{Re}\left(\hat{\phi}_X(\omega_j)\right)\right] \\ &= \frac{\left(\mathrm{Re}\left(\phi_X(\omega_i - \omega_j)\right) + \mathrm{Re}\left(\phi_X(\omega_i + \omega_j)\right) - 2\mathrm{Re}\left(\phi_X(\omega_i)\right)\mathrm{Re}\left(\phi_X(\omega_j)\right)\right)}{2N}\end{aligned} \quad (16b)$$

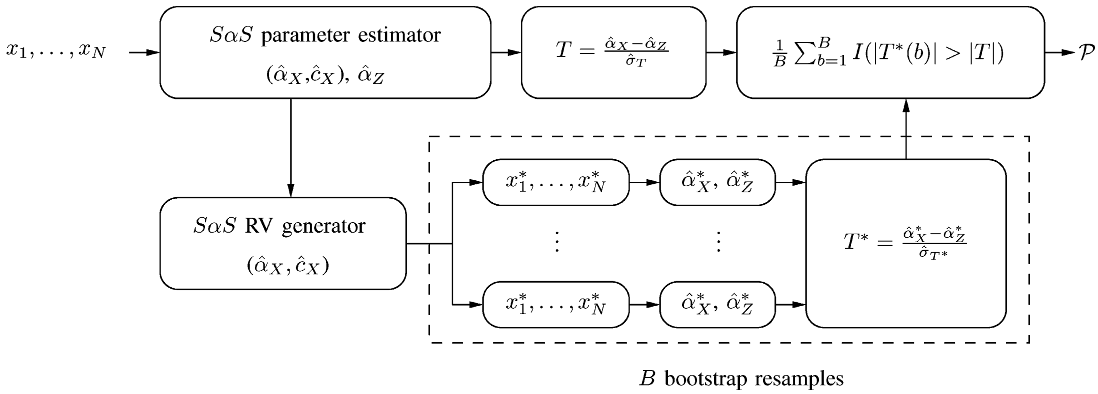

Fig. 2. Parametric bootstrap procedure for estimating the null distribution of the test statistic T and the p-values for the stability test.

$\hat{\alpha}_{Z_2} - \hat{\alpha}_X$, where the segments from which $\hat{\alpha}_X$ and $\hat{\alpha}_{Z_2}$ were estimated do not overlap. $\hat{\alpha}_X$ and $\hat{\alpha}_{Z_2}$ are then independent with distributions

$$\hat{\alpha}_X \overset{a}{\sim} \mathrm{N}\left(\mu_{\alpha_X}(\alpha, c, \boldsymbol{\omega}), \sigma^2_{\alpha_X}(\alpha, c, \boldsymbol{\omega})\right) \tag{24a}$$

$$\hat{\alpha}_{Z_2} \overset{a}{\sim} \mathrm{N}\left(\mu_{\alpha_{Z_2}}(\alpha, c, \boldsymbol{\omega}), \sigma^2_{\alpha_{Z_2}}(\alpha, c, \boldsymbol{\omega})\right) \tag{24b}$$

where $\mathrm{N}(\mu, \sigma^2)$ denotes a normal distribution with mean μ and variance σ^2. Since the estimator is asymptotically unbiased

$$\hat{T}_2 \overset{a}{\sim} \mathrm{N}\left(0, \sigma^2_{\alpha_X} + \sigma^2_{\alpha_{Z_2}}\right). \tag{25}$$

Likewise, the asymptotic distribution of the test statistic $\hat{T}_3 = \hat{\alpha}_{Z_3} - \hat{\alpha}_X$ for nonoverlapping segments is

$$\hat{T}_3 \overset{a}{\sim} \mathrm{N}\left(0, \sigma^2_{\alpha_X} + \sigma^2_{\alpha_{Z_3}}\right). \tag{26}$$

The p-value associated with a test statistic is found from its asymptotic distribution evaluated at estimates of the parameters. For $\hat{T}_2$

$$\mathcal{P}_2 = 2\left(1 - \Phi\left(\frac{|\hat{T}_2|}{\sqrt{\sigma^2_{\alpha_X} + \sigma^2_{\alpha_{Z_2}}}}\right)\right) \tag{27}$$

and likewise for $\hat{T}_3$

$$\mathcal{P}_3 = 2\left(1 - \Phi\left(\frac{|\hat{T}_3|}{\sqrt{\sigma^2_{\alpha_X} + \sigma^2_{\alpha_{Z_3}}}}\right)\right) \tag{28}$$

where $\Phi(\cdot)$ is the standard normal cdf. The hypothesis test is then carried out by testing $T_2 = 0 \cap T_3 = 0$ using a MHT procedure.

B. Bootstrap Estimator

The bootstrap is a resampling based statistical technique suited to the general problem of estimating sampling distributions [35]–[38]. Here, it is used to estimate the null distribution of a test statistic. Although the primary advantage of the bootstrap is that it replaces complex or intractable theoretical analysis with computation, two other factors are important in this application. First, bootstrap estimates of the null distribution may be more accurate than asymptotically derived ones for finite samples. Second, the bootstrap may account for dependence between the estimated characteristic exponents when they are calculated from overlapping samples. This follows from the plug-in-principle of the bootstrap espoused in [35] and [36]. In essence, this principle states that the relationships between sample statistics are generally mirrored in the bootstrap statistics since the mechanism by which they are calculated from a sample is the same in both cases. The mechanism here includes the way in which the sample is separated into segments and the $S\alpha S$ parameter estimator.

The nonparametric bootstrap treats the observations as an estimate of the sampling distribution and resampling involves drawing with replacement from this set. Should the observations arise from a distribution within the domain of attraction of a non-Gaussian αS law, the nonparametric bootstrap does not capture the characteristics of the sampling distribution[36]. In this case, the parametric bootstrap is known to behave correctly [35].

Fig. 2 summarizes the parametric bootstrap procedure for this problem. The parametric bootstrap assumes that given any necessary parameters observations are easily generated under the null hypothesis, estimates of the parameters suffice if they are unknown. This is shown in the two left-most blocks of Fig. 2, where, from the observations $x_1, \ldots, x_N$, the distributional parameters $(\hat{\alpha}_X, \hat{c}_X)$ are found using Koutrouvelis' estimator, and given these, the bootstrap resamples $x_1^*, \ldots, x_N^*$ are generated from a $S\alpha S$ random variable (RV) generator. $S\alpha S$ random variables are generated using the computationally efficient method of Chambers *et al.* [39]. The bootstrap test statistics $T^* = (\hat{\alpha}_X^* - \hat{\alpha}_Z^*)/\hat{\sigma}_{T^*}$ are calculated from the bootstrap resamples in exactly the same way the test statistic $T = (\hat{\alpha}_X - \hat{\alpha}_Z)/\hat{\sigma}_T$ is found from the observations: by finding the difference between estimates of the characteristic exponents. A sufficient number B of these bootstrap statistics form the bootstrap estimate of the null distribution. Guidelines for choosing B can be found in [40], although $B \geq 10/\zeta$ is generally sufficient. Finally, the p-values $\mathcal{P} = (1/B)\sum_{b=1}^{B} I(|T^*(b)| > |T|)$ are obtained by comparing the test statistic to the bootstrapped ones. A similar approach was employed for inference on the parameters of an αS distribution and for a cf domain goodness-of-fit test for αS distributions [8].

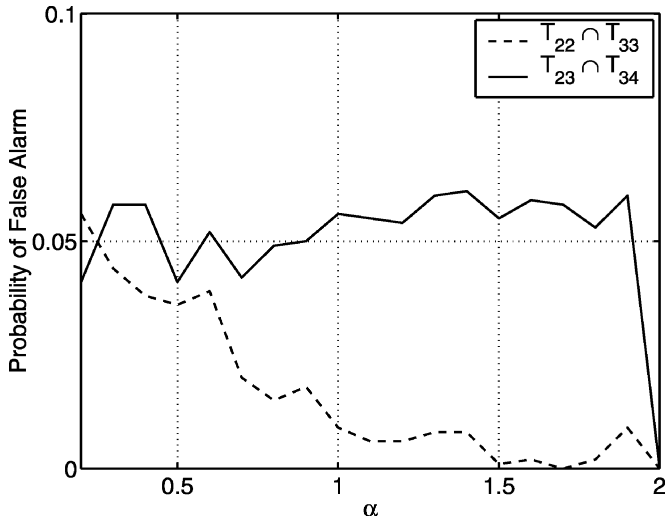

Fig. 3. Probability of false alarm for the stability test using asymptotic statistics. —— denotes $T_{22} \cap T_{33}$, and — denotes $T_{23} \cap T_{34}$.

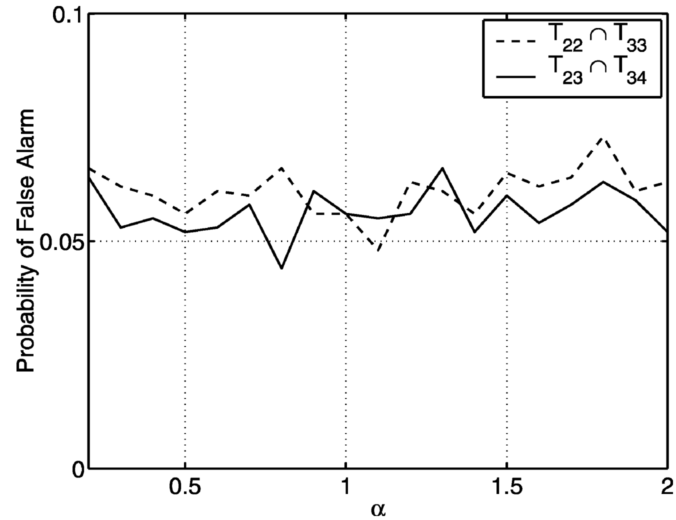

Fig. 4. Probability of false alarm for the stability test using the bootstrap with pivoting. —— denotes $T_{22} \cap T_{33}$, and — denotes $T_{23} \cap T_{34}$.

In Fig. 2, the test statistics and their bootstrapped versions are studentized by dividing by an estimate of the square root of their respective variances $\hat{\sigma}_T$ and $\hat{\sigma}_{T^*}$. This is done to obtain asymptotically pivotal test statistics, whose asymptotic distributions are independent of any unknown parameters. This improves the accuracy of the bootstrap estimates of the null distributions [40]. A nested bootstrap procedure may be used to estimate these variances where another layer of bootstrap resampling is carried out on the resamples $x_1^*, \ldots, x_N^*$, and from these, $\hat{\sigma}_{T^*}$ is obtained. Similarly, bootstrap resampling is carried out on the observations $x_1, \ldots, x_N$ to find $\hat{\sigma}_T$. To reduce the computational complexity, Monte Carlo estimates of σ_T and σ_{T^*} were found offline.

VI. Experiments

Sample sizes of approximately 500 are necessary for the asymptotically derived critical values to be accurate. Since the observations are separated into at most four segments, sample sizes of 2000 were used. The probability of false alarm and power of the test were found from 1000 independent Monte Carlo realizations.

Throughout the experiments, the linear regression estimator of (13) was used where the ecf was sampled at the points $\omega_k = \pi k/25$, $k = 1, \ldots, K$. Using the asymptotic results derived in Section V-A, K was chosen to minimize the asymptotic MSE of $\hat{\alpha}$ [41]. Koutrouvelis determined the ω_k in the same way but used Monte Carlo simulations instead of asymptotic theory. It was found that by minimizing the asymptotic MSE of $\hat{\alpha}$, the variance of $\hat{\alpha}$ was reduced compared with Koutrouvelis' estimator, especially for $\alpha \approx 2$, without resorting to heavy Monte Carlo analysis.

A. Maintenance of the Set Level, Results, and Discussion

Fig. 3 shows the P_{FA} for the stability test using asymptotic statistics. When characteristic exponents are estimated from nonoverlapping segments, as for $T_{23} \cap T_{34}$, the set level is closely maintained. An exception occurs when $\alpha \approx 2$ and the P_{FA} is markedly less than the set level. The cause is a rapid change in the variance of $\hat{\alpha}$ versus α for $\alpha \approx 2$. When $\hat{\alpha}$ is less than the true $\alpha \approx 2$, the estimate of $\mathsf{Var}[\hat{\alpha}]$ is much larger than its true value, making the test more conservative. The effect decreases with sample size, and asymptotically, the set level is maintained at $\alpha = 2$.

When characteristic exponents are estimated from overlapping segments, as for the test statistics T_{22} and T_{33}, the set level is not maintained, the tests being far too conservative. To explain, take $T_{22} = \alpha_{Z_{22}} - \alpha_X$, which has variance $\mathsf{Var}[T_{22}] = \mathsf{Var}[\alpha_{Z_{22}}] + \mathsf{Var}[\alpha_X] - 2\mathsf{Cov}[\alpha_{Z_{22}}, \alpha_X]$. Under the null hypothesis, there is a high degree of positive correlation between the estimates of these two characteristic exponents. Recall that the cause of this dependence was described in Section IV-A, having been the result of determining the characteristic exponents from overlapping segments. The asymptotic estimate for the variance, which does not account for the covariance between the two characteristic exponents, is much larger than the true value, leading to a conservative test. The same is true of T_{33}.

Fig. 4 shows the P_{FA} for the stability test using the bootstrap and pivotal test statistics. The set level is maintained for both overlapping and nonoverlapping segments, showing that the bootstrap accounts for dependence between characteristic exponents estimated from overlapping segments. Although not shown here, nonpivotal statistics produced similar results, except for $\alpha \approx 2$, where it was as low as 2%. This drop is again attributed to the rapid change in variance of $\hat{\alpha}$ in this region. The effect is eliminated by using pivotal statistics.

Empirical critical values for the D and A^2 tests were calculated for $0.5 \leq \alpha \leq 2$ over 10 000 Monte Carlo simulations. For $\alpha < 0.5$, calculation of the critical values became too expensive computationally. Hence, the P_{FA} was only evaluated for $0.6 \leq \alpha \leq 2$ to ensure that $\hat{\alpha} \geq 0.5$. Although not shown here, the tests were found to maintain the set level.

B. Power of the Tests: Results and Discussion

The tests were evaluated under a variety of alternative distributions, including Student's t distribution with 2, 3, 4 and 10 degrees of freedom, denoted t_2, t_3, t_4, and t_{10} respectively: the

TABLE I
POWER OF THE STABILITY TEST USING ASYMPTOTIC STATISTICS, PIVOTAL PARAMETRIC BOOTSTRAP STATISTICS, AND EMPIRICAL DISTRIBUTION FUNCTION (edf) STATISTICS

	Asymptotic		Pivotal Parametric Bootstrap						edf	
	$T_{22} \cap T_{33}$	$T_{23} \cap T_{34}$	T_{22}	T_{33}	T_{23}	T_{34}	$T_{22} \cap T_{33}$	$T_{23} \cap T_{34}$	D	A^2
t_2	0.27	0.22	0.50	0.60	0.16	0.26	0.63	0.27	0.05	0.06
t_3	0.45	0.31	0.71	0.76	0.17	0.28	0.82	0.27	0.05	0.10
t_4	0.45	0.30	0.71	0.79	0.16	0.25	0.83	0.26	0.07	0.09
t_{10}	0.27	0.09	0.40	0.35	0.06	0.07	0.40	0.08	0.07	0.08
$\mathrm{L}(0,1)$	1.00	0.93	1.00	1.00	0.71	0.85	1.00	0.91	0.42	1.00
$\beta(4,4)$	0.30	0.13	0.87	0.89	0.56	0.56	0.95	0.72	1.00	1.00
$\varepsilon-\mathrm{mix}(0.01,10)$	0.01	0.01	0.04	0.04	0.04	0.03	0.04	0.03	0.04	0.05
$\varepsilon-\mathrm{mix}(0.01,100)$	0.02	0.19	0.35	0.46	0.13	0.17	0.44	0.21	0.07	0.06
$\varepsilon-\mathrm{mix}(0.1,10)$	0.07	0.06	0.12	0.30	0.02	0.07	0.26	0.06	0.05	0.06
$\varepsilon-\mathrm{mix}(0.1,100)$	0.98	0.68	1.00	1.00	0.47	0.63	1.00	0.68	0.24	1.00
GM_1	1.00	1.00	1.00	1.00	0.98	1.00	1.00	1.00	1.00	1.00
GM_2	0.73	0.35	0.93	0.52	0.68	0.24	0.93	0.64	1.00	1.00
GM_3	1.00	1.00	1.00	1.00	0.93	1.00	1.00	1.00	1.00	1.00

Laplace distribution with zero mean and unit variance, L(0,1); the β distribution with both parameters equal to 4, β(4,4). Also considered were several Gaussian mixture distributions. The $\varepsilon-\mathrm{mix}(\varepsilon,\kappa)$ distribution with pdf $(1-\varepsilon)\mathrm{N}(0,\sigma^2)+\varepsilon\mathrm{N}(0,\kappa\sigma^2)$, $0<\varepsilon<1$, $\kappa>1$ is a popular model for heavy-tailed observations and approximates Middleton's Class A model [42]. The second component models impulsive noise, where impulsive events occur with probability ε and have a variance κ times greater than the first component that models Gaussian noise with variance σ^2.

Gaussian mixture distributions with pdf $\sum_{k=1}^{K} w_k \exp(-(x-\mu_k)^2/2\sigma_k^2)/\sqrt{2\pi\sigma_k^2}$ were also tested, where the means μ_k, variances σ_k^2, and weights w_k of each component are contained in the vectors $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$, and $\boldsymbol{w}$. Three cases were chosen: GM_1 with $\boldsymbol{\mu}=(-0.1,0.2,0.5,1)^{\mathsf{T}}$, $\boldsymbol{\sigma}=(0.25,0.4,0.9,1.6)^{\mathsf{T}}$, $\boldsymbol{w}=(0.5,0.1,0.2,0.2)^{\mathsf{T}}$; GM_2 with $\boldsymbol{\mu}=(-0.3,0.7,0.8)^{\mathsf{T}}$, $\boldsymbol{\sigma}=(0.5,0.25,0.4)^{\mathsf{T}}$, $\boldsymbol{w}=(0.2,0.5,0.3)^{\mathsf{T}}$; GM_3 with $\boldsymbol{\mu}=(-0.2,0,0.2,0.3)^{\mathsf{T}}$, $\boldsymbol{\sigma}=(0.25,0.4,0.9,1.6)^{\mathsf{T}}$; and $\boldsymbol{w}=(0.5,0.1,0.2,0.2)^{\mathsf{T}}$. All are slightly skewed and, except for GM_3, nonzero mean alternatives.

Table I shows the power of the stability and empirical distribution function tests. Note that the stability test that uses overlapping segments and asymptotic theory cannot be compared with the others fairly as it is conservative and does not maintain the set level.

The stability test using the bootstrap is generally the most powerful, and it is able to reject symmetric heavy-tailed alternatives that are very difficult to distinguish from $S\alpha S$ distributions. These include the t and ε-mix distributions. Experiments with nonpivotal statistics showed that pivoting does not have a significant effect on the power of the tests. Concerning the use of the individual test statistics, it was found that from most to least powerful, they were, in general, T_{33}, T_{22}, T_{34}, and T_{23}.

It is expected that test statistics formed from overlapping samples (T_{22} and T_{33}) are more powerful than those formed from nonoverlapping samples (T_{23} and T_{34}). The larger number of samples in each segment reduces the error in the characteristic exponent estimates, leading to a more distinct separation between the test statistics under the null and alternative hypotheses.

That the power of test statistics formed from summing three segments exceeds that of those formed from two segments has no simple explanation, but an intuitive reason is offered. Consider what happens as the number of segments grows very large, but the number of samples in each stays constant. Assuming that the sum of these segments converges to a limiting distribution, this limiting distribution is αS by the generalized CLT. As the number of segments increases, the difference in power between a test that uses M and $M+1$ segments will diminish. Since, in any practical implementation of the test, the number of samples in each segment decreases as the number of segments increase, the power of the test is not likely to increase and may drop. Although this suggests using fewer segments, this must be offset against the possibility that summing more segments leads to a distribution and, hence, a characteristic exponent, which is further from the original than for fewer. The observed powers suggest that in the majority of cases, the trade off is in favor of three segments. Taking these influences into account, it is possible that by using four or more segments, the power may be further increased.

VII. Conclusion

A test for $S\alpha S$ distributions was developed based on their unique stability property. Null distributions were derived using asymptotic theory and estimated with the parametric bootstrap. The bootstrap technique maintained the set level of the test while achieving high power in detecting alternatives that possessed very similar tail behavior to $S\alpha S$ distributions, compared with empirical distribution function tests. Several straightforward improvements are possible.

The power of the test depends on the accuracy with which the characteristic exponent can be estimated. More accurate estimators, such as maximum likelihood, will increase the power at the cost of computational complexity.

The stability property holds for nonsymmetric αS distributions if the skewness parameter β is included. By testing for the equality of both α and β, an αS goodness-of-fit test is obtained.

Multivariate αS distributions possess a complex structure where the skewness and scale parameters are combined into a finite measure on the unit sphere. A general method for obtaining goodness-of-fit tests for multivariate αS distributions from univariate tests was suggested in [16] and is based on the following theorem (see [10, Th. 2.1.5]). Let $\boldsymbol{X}$ be a d-dimensional random vector in $\mathbb{R}^d$. $\boldsymbol{X}$ is a 1) strictly αS, 2) $S\alpha S$, and 3) αS random vector in $\mathbb{R}^d$ if all linear combinations of the components of $\boldsymbol{X}$ are 1) strictly αS, 2) $S\alpha S$, and 3) αS with $\alpha \geq 1$. If follows that to test for either of these three cases, every linear combination must be assessed using a univariate test. In practice, only a finite number of directions along which to project the random vector are chosen. An advantage is that only univariate tests and estimators are required, which avoids the need for more complicated multivariate estimators [43], [44].

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